Differential Geometry: Reflections generate an isometry

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Theorem 0.1. *An isometry of $\mathbb{R}^n$ is product (composition) of atmost $n+1$ reflections.*

Proof. We give proof by induction For $n = 2$ we know that the orthogonal transformations of $\mathbb{R}^2$ are described as either

$$\rho_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

which are rotation by angle θ in anti-clockwise direction and

$$R_\theta = \begin{pmatrix} -\cos\theta & \sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

which are reflections with respect to the line $\mathbb{R}(\cos\theta/2, \sin\theta/2)$ (verify).

Given an isometry i of $\mathbb{R}^2$, let $i(0) = v$ and T be a reflection of $\mathbb{R}^2$ which maps v to 0 (Ex. Write explicit expression for such T). Then $T \circ i$ is an isometry of $\mathbb{R}^2$ which fixes origin and hence is an orthogonal transformation. Therefore $T \circ i = \rho_\theta$ or $T \circ i = R_\theta$ for some θ. Since a rotation is composition of two reflections $\rho_\theta = R_{2\theta} \circ R_\theta$, we get either $i = T \circ R_{2\theta} \circ R_\theta$ or $i = T \circ R_\theta$ and result is true for $\mathbb{R}^2$.

By induction, assume that an isometry of $\mathbb{R}^k$ can be written as composition of atmost $k+1$ reflections for any $k \leq n$. Let $i : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ be an isometry and $\{e_1, \ldots, e_{n+1}\}$ denote the standard basis of $\mathbb{R}^{n+1}$. Let T be a reflection of $\mathbb{R}^{n+1}$ which maps $i(e_{n+1})$ to e_{n+1}. Then $T \circ i$ is an isometry of $\mathbb{R}^{n+1}$ which fixes the basis vector e_{n+1}.

Claim: $T \circ i \mid_{\mathbb{R}^n} : \mathbb{R}^n \to \mathbb{R}^n$ is an isometry of $\mathbb{R}^n$.
Proof of Claim : Ex.

Thus $T \circ i \mid_{\mathbb{R}^n}$ is a product of atmost $n+1$ reflections, say $T \circ i \mid_{\mathbb{R}^n} = R_1 \circ \ldots \circ R_k$, $k \leq n+1$. Note that any reflection R_i of $\mathbb{R}^n$ can be extended to a reflection of $\mathbb{R}^{n+1}$ by defining

$$\tilde{R}_i(v_1, \ldots, v_n, v_{n+1}) = (R_i(v_1, \ldots, v_n), v_{n+1})$$

For if $R : \mathbb{R}^n \to \mathbb{R}^n$ is a reflection with respect to a hyperplane W with $u \in \mathbb{R}^n$ normal to W defined by

$$R(v) = v - 2 < v, u > u$$

then defining $\tilde{R} : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ as

$$\tilde{R}w = Rw' + w_0 e_{n+1}$$

where we write $\mathbb{R}^{n+1} = \mathbb{R}^n \oplus \mathbb{R}$ and $w \in \mathbb{R}^{n+1}$ is written as $w = w' + w_0 e_{n+1}$ with $w' \in \mathbb{R}^n$, $w_0 \in \mathbb{R}$. Then $\tilde{R}w = Rw' + w_0 e_{n+1} = w' - 2 < w', u > u + w_0 e_{n+1} = w - 2 < w, u > u$ is reflection with respect to the hyperplane $W \oplus \mathbb{R}$. Here u sits in $\mathbb{R}^{n+1}$ as $u + 0e_{n+1}$.
Thus, $i = T \circ \tilde{R}_1 \circ \ldots \circ \tilde{R}_k$, $k \leq n + 1$ and the proof is complete.

$\square$